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ON THE LIMIT OF REPEATED ADJUSTMENTS.

BY E. L. DE FOREST, M. A., WATERTOWN, CONN.

IT has been shown by me (ANALYST, May 1878, p. 70) that when a series is adjusted by a given formula several times in succession, the process is precisely equivalent to a single adjustment made by a certain resultant formula, and that the coefficients in such resultant formulas, considered as terms in a series or ordinates to a curve, follow outlines which appear to resemble each other, no matter what the outline of the original formula may have been.

We will now investigate the form of the curve followed by such coefficients at the limit, when the number of repetitions becomes very large, or infinite.

Proposition. If a series is adjusted k times in succession by the formula

$$u_0' = \lambda_0 u_0 + \lambda_1(u_1 + u_{-1}) + \lambda_2(u_2 + u_{-2}) + \ \ldots\ldots + \lambda_m(u_m + u_{-m}), \qquad (1)$$

and if the equivalent or resultant formula is denoted by

$$u_0^{(k)} = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + \ \ldots + l_{km}(u_{km} + u_{-km}), \qquad (2)$$

then any coefficient l_i and the $2m$ other coefficients nearest to it will be connected by the relation

$$\frac{\lambda_1(l_{i+1} - l_{i-1}) + 2\lambda_2(l_{i+2} - l_{i-2}) + \ \ldots\ldots + m\lambda_m(l_{i+m} - l_{i-m})}{\lambda_0 l_i + \lambda_1(l_{i+1} + l_{i-1}) + \lambda_2(l_{i+2} + l_{i-2}) + \ \ldots + \lambda_m(l_{i+m} + l_{i-m})} = \frac{-i}{k+1}. \qquad (3)$$

This property has been reached inductively, by proceeding from some very simple cases to more complex ones, and I am unable to offer any general proof of it, but it has been tested in such a variety of ways that there can be no doubt of its generality. Take for example

$$u_0' = \tfrac{1}{21}[7u_0 + 6(u_1 + u_{-1}) + 3(u_2 + u_{-2}) - 2(u_3 + u_{-3})].$$

By the method given in the ANALYST referred to, it is found that three successive applications of this formula are exactly equivalent to

$$u_0''' = \frac{1}{9261}[2617u_0+2232(u_1+u_{-1})+1278(u_2+u_{-2})+276(u_3+u_{-3}) \\ -171(u_4+u_{-4})-234(u_5+u_{-5})-105(u_6+u_{-6})+\ 18(u_7+u_{-7}) \\ +\ 36(u_8+u_{-8})-\ \ 8(u_9+u_{-9})].$$

Now take, for instance, $i=2$, and (3) becomes

$$\frac{6(276-2232)+2\times3(-171-2617)+3\times\ \ 2(-234-2232)}{7\times1278+6(276+2232)+3(-171+2617)-2(-234+2232)} = \frac{-2}{3+1},$$

in which both members reduce to the same value $-\frac{1}{2}$. By such trials, any one can satisfy himself that formula (3) is true, no matter what the values of i, k, λ_0, λ_1, &c. may be. If any l beyond l_{km} should be included, its value is zero, and any negative sub-index of l is the same as positive. The resultant coefficients shown in Tables I. and II. of the ANALYST article cited, will not afford exact tests of the present proposition, because they are only correct to three places of decimals.

Now let Δ_1, Δ_2, &c. denote the finite differences of the series l_i, l_{i+1}, &c., so that we have

$$l_{i+1}-l_i = \Delta_1, \qquad l_{i+3}-3l_{i+2}+3l_{i+1}-l_i = \Delta_3,$$
$$l_{i+2}-2l_{i+1}+l_i = \Delta_2, \qquad l_{i+4}-4l_{i+3}+6l_{i+2}-4l_{i+1}+l_i = \Delta_4,$$
$$l_{i+5}-5l_{i+4}+10l_{i+3}-10l_{i+2}+5l_{i+1}-l_i = \Delta_5,$$

and for convenience let differences higher than Δ_5 be neglected, so that Δ_5 is regarded as constant for any six consecutive coefficients l. Thus we get

$$\begin{array}{ll} l_{i+1}=l_i+\ \Delta_1, & l_{i-1} = l_i-\ \Delta_1+\ \ \Delta_2-\ \ \Delta_3+\ \ \Delta_4-\ \ \Delta_5, \\ l_{i+2}=l_i+2\Delta_1+\ \Delta_2, & l_{i-2} = l_i-2\Delta_1+\ 3\Delta_2-\ 4\Delta_3+\ 5\Delta_4-\ 6\Delta_5, \\ l_{i+3}=l_i+3\Delta_1+3\Delta_2+\ \Delta_3, & l_{i-3} = l_i-3\Delta_1+\ 6\Delta_2-10\Delta_3+15\Delta_4-21\Delta_5, \\ l_{i+4}=l_i+4\Delta_1+6\Delta_2+4\Delta_3+\Delta_4, & l_{i-4} = l_i-4\Delta_1+10\Delta_2-20\Delta_3+35\Delta_4-56\Delta_5, \\ \text{&c.} & \text{&c.} \end{array}$$

By subtracting and adding, we find in general

$$l_{i+n}-l_{i-n}=2n\Delta_1-n\Delta_2+\tfrac{1}{3}n(n^2+2)\Delta_3-\tfrac{1}{2}n(n^2+1)\Delta_4+\tfrac{1}{60}n(n^4+35n^2+24)\Delta_5,$$
$$l_{i+n}+l_{i-n}=2l_i+n^2(\Delta_2-\Delta_3)+\tfrac{1}{12}n^2(n^2+11)\Delta_4-\tfrac{1}{6}n^2(n^2+5)\Delta_5.$$

These expressions would remain the same if differences higher than Δ_5 had been taken into account, only terms containing Δ_6, Δ_7, &c., would be added.

Substitute the above in (3), and write, for brevity,

$$\left.\begin{array}{l} \lambda_0\ \ \ +2(\lambda_1\ +\lambda_2+\ldots\ldots+\lambda_m)= b \\ 1^2\lambda_1+2^2\lambda_2+\ldots\ldots+m^2\lambda_m = b_2 \\ 1^4\lambda_1+2^4\lambda_2+\ldots\ldots+m^4\lambda_m = b_4 \\ 1^6\lambda_1+2^6\lambda_2+\ldots\ldots+m^6\lambda_m = b_6 \\ \qquad \text{&c.} \qquad\qquad \text{&c.} \end{array}\right\} \tag{4}$$

then (3) becomes

$$\frac{b_2(2\Delta_1-\Delta_2)+\frac{1}{3}(2b_2+b_4)\Delta_3-\frac{1}{2}(b_2+b_4)\Delta_4+\frac{1}{60}(24b_2+35b_4+b_6)\Delta_5-\text{&c.}}{bl_i+b_2(\Delta_2-\Delta_3)+\frac{1}{12}(11b_2+b_4)\Delta_4-\frac{1}{6}(5b_2+b_4)\Delta_5+\text{&c.}}$$
$$= \frac{-i}{k+1}. \tag{5}$$

But if (1) is an adjustment formula of such nature as to give exact results when applied to a series of the first order, that is, one whose first differences are constant, we shall have $b = 1$. If it is exact in the case of a series of the third order, we shall have both $b = 1$ and $b_2 = 0$. If exact for a series of the fifth order, then $b=1$, $b_2 = 0$ and $b_4 = 0$; and so on for higher orders. This has been shown by Schiaparelli in his essay on empirical curves, *Sul modo di ricavare*, etc., as remarked by me in the *Smithsonian Reports* of 1871 and 1873, pp. 335 and 342. It can also be proved somewhat differently, as follows. Equidistant ordinates to the curve

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.,$$

form a series of an order equal to the degree of the curve, and terms in such series can always be expressed by such an equation, the origin being taken at any term we please, while the constant interval between the terms may be the unit of x. Thus we have $y_0 = A_0$, and

$$\begin{array}{ll} y_1 = A_0 + A_1 + A_2 + \&c., & y_{-1} = A_0 - A_1 + A_2 - A_3 + \&c., \\ y_2 = A_0 + 2A_1 + 2^2 A_2 + 2^3 A_3 + \&c., & y_{-2} = A_0 - 2A_1 + 2^2 A_2 - 2^3 A_3 + \&c., \\ y_3 = A_0 + 3A_1 + 3^2 A_2 + 3^3 A_3 + \&c., & y_{-3} = A_0 - 3A_1 + 3^2 A_2 - 3^3 A_3 + \&c., \\ \&c., & \&c., \end{array}$$

and the adjustment formula

$$y'_0 = \lambda_0 y_0 + \lambda_1 (y_1 + y_{-1}) + \lambda_2 (y_2 + y_{-2}) + \ldots + \lambda_m (y_m + y_{-m})$$

becomes by substitution

$$\begin{aligned} y'_0 = {} & A_0[\lambda_0 + 2(\lambda_1 + \lambda_2 + \ldots + \lambda_m)] + 2A_2(1^2\lambda_1 + 2^2\lambda_2 + \ldots + m^2\lambda_m) \\ & + 2A_4(1^4\lambda_1 + 2^4\lambda_2 + \ldots + m^4\lambda_m) + 2A_6(1^6\lambda_1 + 2^6\lambda_2 + \ldots + m^6\lambda_m) + \&c., \end{aligned}$$

which by the notation of (4) is equivalent to

$$y'_0 = A_0 b + 2(A_2 b_2 + A_4 b_4 + A_6 b_6 + \&c.).$$

In the case of a series of the first order, A_2, A_4 &c. will be zero, and we cannot have $y'_0 = y_0 = A_0$ without having $b = 1$. For a series of the second or third order, A_4, A_6, &c. will be zero, and since A_0 and A_2 may be supposed to have any values, we cannot have $y'_0 = y_0$ without having both $b = 1$ and $b_2 = 0$. Likewise for a series of the fourth or fifth order, we must have $b = 1$, $b_2 = 0$, $b_4 = 0$; and so on for series of higher orders.

Reverting now to equation (5), let us see what becomes of it in each of the foregoing cases at the limit, when the number of repetitions of the adjusting process is very great, so that k is a very large number, or infinite. The successive values of l become consecutive ordinates to the limiting curve, so that $l_t = y$, $\Delta_1 = dy$, $\Delta_2 = d^2y$, $\Delta_3 = d^3y$, &c.

Differentials of higher orders are to be neglected in comparison with those of a lower order, and all differentials are neglected in comparison with y.

Taking l_0 as the axis of Y, the abscissa corresponding to any y will be $x = i\Delta x$, which becomes $x = idx$ at the limit. Thus in the first case, with only the condition $b = 1$, (5) reduces to

$$\frac{2b_2 dy}{y} = \frac{-x}{(k+1)dx}, \qquad \therefore \frac{dy}{dx} = \frac{-\frac{1}{2}xy}{b_2(k+1)(dx)^2}. \tag{6}$$

With the two conditions $b = 1$, and $b_2 = 0$, (5) becomes

$$\frac{\frac{1}{3}b_4 d^3y}{y} = \frac{-x}{(k+1)dx}, \qquad \therefore \frac{d^3y}{dx^3} = \frac{-3xy}{b_4(k+1)(dx)^4}, \tag{7}$$

and with the three conditions $b = 1$, $b_2 = 0$, $b_4 = 0$, (5) is

$$\frac{\frac{1}{60}b_6 d^5y}{y} = \frac{-x}{(k+1)dx}, \qquad \therefore \frac{d^5y}{dx^5} = \frac{-60xy}{b_6(k+1)(dx)^6}. \tag{8}$$

It is to be presumed that if orders of differences higher than the fifth were taken into account, four or more of the conditions would give differential equations of the seventh and higher orders similar to the above. We may say then in general, that the limiting curve sought will be characterized by the linear differential equation

$$\frac{d^n y}{dx^n} = axy, \tag{9}$$

where n is an odd number, and has the values 1, 3, 5, &c., according as the original adjustment formula is exact in the case of series of the first, third, fifth, &c. orders. The simplest case, $n = 1$, gives

$$\frac{dy}{y} = axdx, \qquad \therefore y = Ce^{\frac{1}{2}ax^2}.$$

The constant of integration C can be determined by the condition $b = 1$, for this is satisfied not only by the original formula (1), but also by the resultant formula for any number of repetitions. (ANALYST, p. 66.) At the limit, the condition becomes

$$\int_{-\infty}^{\infty} ydx = 1, \qquad \therefore C = \sqrt{\frac{-a}{2\pi}}.$$

Formula (6) shows that a is, in general, essentially negative. Writing $a = -2h^2$, we get

$$h = \frac{1}{2dx\sqrt{[b_2(k+1)]}}, \qquad y = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}, \tag{10}$$

the well known equation of the probability curve, which is thus seen to be the limiting form of the coefficients, for adjustment formulas which satisfy the single condition $b = 1$. Such, for instance, are the simple mean

$$u'_0 = \frac{1}{2m+1}\left\{ u_0 + (u_1 + u_{-1}) + (u_2 + u_{-2}) + \ldots + (u_m + u_{-m}) \right\},$$

and this other which has some times been used,

$$u'_0 = \frac{1}{25}\left\{ 5u_0+4(u_1+u_{-1})+3(u_2+u_{-2})+2(u_3+u_{-3})+(u_4+u_{-4}) \right\}.$$

All such formulas give exact results only when applied to arithmetical progressions, that is, to series which follow not a curve, but a straight line.

Since the differentials of x and y bear to each other nearly the same relations as their finite differences do, provided these are small, it follows that the equation (10) expresses pretty nearly the outline of the resultant formula for any finite, but tolerably large, number of repetitions, and the larger the number, the more accurate is the representation. Take for example sixteen applications of

$$u'_0 = \tfrac{1}{3}(u_0+u_1+u_{-1}), \tag{11}$$

for which $b_2 = 1^2\lambda_1 = \frac{1}{3}$ and $k+1 = 17$. For dx put $\Delta x = 1$. Then

$$h = .21004, \qquad \log y = \bar{1}.07374-.019160x^2,$$

and Table I. shows the values of y for successive values of x.

TABLE I. ($h = .21004.$)

x	0	1	2	3	4	5	6	7	8	9	10	11	&c.
y	.119	.113	.099	.080	.059	.039	.024	.014	.007	.003	.001	.001	&c.

They come quite near to the true coefficients .121, .115, .101, .081, .059, .039, .023, .012, .006, .003, .001, .000, &c., as found by repeated use of formula (7), ANALYST, p. 67.

Instead of y, it would have been more strictly correct to use the finite difference of the area, having $\Delta x = 1$ for its base and y for its middle ordinate; but the numerical value of such a portion of the area is very nearly the same as that of y.

As is well known, all probability curves are similar figures in this sense, that any increase of the probable error gives a proportionate increase to all the abscissas, while all the ordinates are diminished in the same ratio, so that the whole area of the curve remains constantly unity. The probable error varies inversely as h and $\sqrt{-a}$, and is shown by (10) to vary directly as $\sqrt{b_2}$ and $\sqrt{(k+1)}$. Hence we can find approximate values of y for any number of repetitions of any formula, from similar values which have already been found for some other number of repetitions of some other formula. Take, for example, eight applications of

$$u'_0 = \tfrac{1}{5}(u_0+u_1+u_{-1}+n_2+u_{-2}), \tag{12}$$

where $k+1 = 9$ and $b_2 = 1^2\lambda_1+2^2\lambda_2 = \frac{1}{5}+\frac{4}{5} = 1$, so that $h = \frac{1}{6}$. Let each x in Table I. be multiplied by the number $\frac{1}{6}\div.21004=.7935$, and let each corresponding y be found by interpolation in that table, thus;

x	0	.79	1.59	2.38	3.17	3.97	4.76	5.55	6.35	7.14	7.93	8.73	9.52	10.3
y	.119	.114	.105	.092	.076	.058	.044	.031	.020	.013	.007	.004	.002	.001

Multipying each y here by .7935 we get

.094 .091 .083 .073 .060 .046 .035 .025 .016 .010 .006 .003 .002 .001,

which is not far from the true series

.098 .095 .087 .075 .061 .047 .034 .023 .014 .008 .004 .002 .001 .000,

and the approximation would be closer if k were greater. If k were infinite all the terms of such a series would be infinitesimals, but the relation between them would be of the form (10).

We come now to the case of chief interest, that in which the adjustment formula satisfies the two conditions $b=1$ and $b_2=0$, so that it gives exact results when applied to a series of the third or any lower order, and is adapted to the adjustment of series which follow not only straight lines, but curved ones. Here (7) gives for the differential equation of the limiting curve

$$a=\frac{-3}{b_4(k+1)(dx)^4}, \qquad \frac{d^3y}{dx^3}=axy. \tag{13}$$

The integration of equations of this kind, under the general form (9), is treated of in various text-books; see for instance Price's *Calculus*, Vol. II., pp. 652—6; but I am not aware that it has ever been discussed under the peculiar conditions imposed by our present problem. The simplest way of getting the value of y, and perhaps the only way adapted for numerical computation, appears to be by means of a series, using the method of inde terminate coefficients. Since the curve is symmetrical on both sides of the middle ordinate or axis of Y, we take only even powers of x, and assume

$$y=A+Bx^2+Cx^4+Dx^6+Ex^8+\&c.,$$

which gives

$$\frac{d^3y}{dx^3}=2.3.4Cx+4.5.6Dx^3+6.7.8Ex^5+\&c,$$

$$axy=Aax+Bax^3+Cax^5+\&c.$$

These are equal to each other, so that

$$(2.3.4C-Aa)x+(4.5.6D-Ba)x^3+(6.7.8E-Ca)x^5+(8.9.10F-Da)x^7 +\&c.=0.$$

The coefficients of each power of x being separately zero, give

$$C=\frac{Aa}{2.3.4}, \qquad D=\frac{Ba}{4.5.6},$$

$$E=\frac{Ca}{6.7.8}=\frac{Aa^2}{2.3.4.6.7.8}, \qquad F=\frac{Da}{8.9.10}=\frac{Ba^2}{4.5.6.8.9.10},$$

$$\&c., \qquad \&c.,$$

and the equation of the curve is

$$y = A\left[1+\frac{ax^4}{2.3.4}+\frac{(ax^4)^2}{2.3.4.6.7.8}+\frac{(ax^4)^3}{2.3.4.6.7.8.10.11.12}+\&c.\right]$$

$$+Bx^2\left[1+\frac{ax^4}{4.5.6}+\frac{(ax^4)^2}{4.5.6.8.9.10}+\frac{(ax^4)^3}{4.5.6.8.9.10.12.13.14}+\&c.\right]. \qquad (14)$$

We may write it thus

$$y = A[1+H_1(ax^4)+H_2(ax^4)^2+\&c.]+Bx^2[1+K_1(ax^4)+K_2(ax^4)^2+\&c.],$$

and exhibit in Table II. the logarithms of H_n and K_n for the first 14 terms of each series.

TABLE II.

n	log H_n.	log K_n.	n	log H_n.	log K_n.
1	$\overline{2}.6167887$	$\overline{3}.9208187$	8	$\overline{28}.7599903$	$\overline{30}.9337.55$
2	$\overline{4}.0934493$	$\overline{5}.0634861$	9	$\overline{32}.1281410$	$\overline{34}.2294878$
3	$\overline{8}.9728755$	$\overline{9}.7242336$	10	$\overline{37}.3552326$	$\overline{39}.3913946$
4	$\overline{11}.4465362$	$\overline{12}.0343922$	11	$\overline{42}.4550623$	$\overline{44}.4319716$
5	$\overline{15}\ 6114802$	$\overline{16}.0687204$	12	$\overline{47}.4389654$	$\overline{49}.3615644$
6	$\overline{19}.5271184$	$\overline{21}.8755956$	13	$\overline{52}.3164218$	$\overline{54}.1888914$
7	$\overline{23}.2336233$	$\overline{25}.4889183$	14	$\overline{57}.0954772$	$\overline{60}.9214004$

At first sight, equation (14) seems to contain three arbitrary constants, but really there is only one, for the two conditions $b = 1$ and $b_2 = 0$ are satisfied by the resultant formula as much as by the original one, and can be used to find A and B in terms of a, as the single condition $b = 1$ was used to find the constant of integration for the probability curve in terms of h. At the limit, (4) shows that the two conditions become

$$\int_0^\infty y\,dx = \tfrac{1}{2}, \qquad \int_0^\infty x^2 y\,dx = 0.$$

But it is not necessary to integrate up to $x = \infty$, for we have seen that for very moderate values of x the value of y becomes so small as to be hardly sensible. (ANALYST, p. 68.)

Let x_1 be the finite limit employed, and integration gives

$$\int_0^{x_1} y\,dx = Ax_1[1+\tfrac{1}{5}H_1(ax_1^4)+\tfrac{1}{9}H_2(ax_1^4)^2+\tfrac{1}{13}H_3(ax_1^4)^3+\&c.]$$

$$+\ Bx_1^3[\tfrac{1}{3}+\tfrac{1}{7}K_1(ax_1^4)+\tfrac{1}{11}K_2(ax_1^4)^2+\&c.], \qquad (15)$$

$$\frac{1}{x_1^3}\int_0^{x_1} x^2 y\,dx = A\,[\tfrac{1}{3}+\tfrac{1}{7}H_1(ax_1^4)+\tfrac{1}{11}H_2(ax_1^4)^2+\&c.]$$

$$+Bx_1^2[\tfrac{1}{5}+\tfrac{1}{9}K_1(ax_1^4)+\tfrac{1}{13}K_2(ax_1^4)^2+\&c.]. \qquad (16)$$

Denoting the sums of the series by S_1, S_2, S_3, S_4, we have

$$Ax_1S_1+Bx_1^3S_2 = \tfrac{1}{2}, \qquad AS_3+Bx_1^2S_4 = 0,$$

and consequently

$$A = \frac{S_4}{2x_1(S_1S_4-S_2S_3)}, \qquad B = -\left(\frac{S_3}{S_4}\right)\frac{A}{x_1^2}. \tag{17}$$

With any given value of a, assuming some sufficiently large value of x_1, we can compute S_1, S_2, &c., and consequently A and B. Any increased value of x_1, up to infinity, ought to give sensibly the same values of A and B. Since S_1, S_2, &c. are functions of ax_1^4, they remain constant so long as ax_1^4 is constant, though a may diminish and x_1^4 increase. Take $ax_1^4 = c$,

$$\therefore\ x_1 = \left(\frac{c}{a}\right)^{\frac{1}{4}},$$

and (17) becomes

$$A = \frac{S_4a^{1/4}}{2(S_1S_4-S_2S_3)c^{1/4}}, \qquad B = -\left(\frac{S_3}{S_4\sqrt{c}}\right)A\sqrt{a}. \tag{18}$$

It thus appears that A and B vary as $a^{1/4}$ and $a^{3/4}$ respectively. Let us assume a sufficiently large value of c, and compute A and B in terms of a. This practical difficulty arises, that the expression for A contains in the denominator the difference of two large numbers, S_1S_4 and S_2S_3, and they are so nearly equal, that if computed by logarithms to seven figures, their difference contains only one or two figures, and the value of A is not found correctly enough. Another computation, therefore, has been made without using logarithms, taking $c = 10000$, and the work was carried to twelve figures. It gave

$$A = .40803a^{1/4}, \qquad B = -.3379891A\sqrt{a}.$$

This value of B is the same as found by logarithms with smaller values of c, while A is a little different, so that the ratio of A to B is known more accurately than the absolute value of A. Putting $a = .001$, and calculating the area of the curve up to a certain point, $x = 36$, and allowing for the area of the rest by the help of the 32-fold coefficients (ANALYST, p. 68), carried to five figures, I found for the total area .99896 instead of the proper value unity. The above value of A, therefore, ought to be increased in the ratio of .99896 to 1, so that the values finally adopted are

$$A = .40845a^{1/4}, \qquad B = -.3379891A\sqrt{a}. \tag{19}$$

The ordinates to the curve, as computed by (14) and (19), are shown in Table III. to four places of decimals.

From what we know of resultant formulas, as discussed in the ANALYST already cited, it is evident that the present curve consists of an infinite number of undulations, lying alternately above and below the axis of X.

TABLE III. ($a=\frac{1}{1000}$.)

x	y	x	y	x	y	x	y
0	.0726	10	.0196	20	—.0068	30	.0010
1	.0719	11	.0133	21	—.0059	31	.0010
2	.0696	12	.0077	22	—.0048	32	.0009
3	.0659	13	.0029	23	—.0037	33	.0008
4	.0610	14	—.0009	24	—.0026	34	.0006
5	.0550	15	—.0038	25	—.0016	35	.0004
6	.0483	16	—.0058	26	—.0007	36	.0002
7	.0412	17	—.0070	27	—.0000	37	&c.
8	.0338	18	—.0075	28	.0005	38	
9	.0265	19	—.0074	29	.0008	39	

Only three of these appear in our table, because as x increases, y becomes so very small as to be hardly sensible, and is only computed with great labor, since (14) gives it as the difference between two large numbers. If plotted on paper, the curve beyond these limits would not be distinguishable by the eye from the axis of X. It presents striking analogies to the probability curve. Since it has only one arbitrary constant a, and its area is always unity, the curves for different values of a are similar figures in the same sense as all probability curves are similar. The dimensions in the direction of x vary inversely as those in the direction of y, and since the latter vary as A and therefore as $a^{1/4}$, it follows from (13) that the x dimensions vary directly as the fourth root of $-b_4$, and also as the fourth root of $k+1$.

Table III. can be used as Table I. was, to compute approximately the coefficients for any given number of repetitions of any given adjustment formula. Take for example the 32-fold use of the formula (ANALYST, p.67),

$$u'_0=\tfrac{1}{35}[17u_0+12(u_1+u_{-1})-3(u_2+u_{-2})],$$

where we have from (4),

$$b_4=1^4\lambda_1+2^4\lambda_2=\tfrac{12}{35}+16\times\tfrac{-3}{35}=-\tfrac{36}{35},$$

while $k=32$ and for dx we put $\Delta x=1$, so that (13) gives $a=\frac{35}{396}$. When $x=0$, then $y=A$, and we have seen that A varies as the fourth root of a, so that any coefficient in the formula now sought may be found by taking the corresponding y from Table III., and multiplying it by the ratio

$$\left(\frac{35}{396}\div\frac{1}{1000}\right)^{\frac{1}{4}}=3.066.$$

The abscissas, found by multiplying this ratio into 0, 1, 2, 3, &c., are

0, 3.07, 6.13, 9.20, 12.26, 15.33, 18.40, 21.46 &c.

The corresponding ordinates, by interpolation in Table III., are

.0726, .0656, .0474, .0251, .0064, —.0045, —.0075, —.0054 &c.

and multiplying these by 3.066, we have
.223, .201, .145, .077, .020, —.014, —.023 —.017, —.006, .001, .003, .002, .000, &c.,
which approach pretty nearly to the true coefficients for 32 applications (ANALYST, p. 68),
.230, .205, .145, .072, .014, —.017, —.022, —.014, —.004, .002, .003 .001, .000, &c.,
and the agreement would be closer if k were a greater number. Taking $k = 64$, we get .188 for the first coefficient, the true value being .192. If k were infinite, all the coefficients would be infinitesimals, but the relation between them would be of the form (14).

By a process quite similar to the foregoing, we can discuss the third case, in which the adjustment formula satisfies the three conditions

$$b = 1, \qquad b_2 = 0, \qquad b_4 = 0,$$

and gives exact results when applied to series of the fifth or any lower order. Such a formula, for instance, is the one at p. 292 of the *Smithsonian Report* of 1871,

$$u'_0 = \tfrac{1}{35}[15(u_0+u_1+u_{-1})-6(u_2+u_{-2})+(u_3+u_{-3})].$$

For an infinite number of repetitions, the differential equation of the limiting curve is given by (8),

$$a = \frac{-60}{b_6(k+1)(dx)^6}, \qquad \frac{d^5y}{dx^5} = axy, \tag{20}$$

and by the method of indeterminate coefficients we find

$$\begin{aligned} y = A\Big(1 &+ \frac{ax^6}{2.3.4.5.6} + \frac{(ax^6)^2}{2.3.4.5.6.8.9.10.11.12} \\ &+ \frac{(ax^6)^3}{2.3.4.5.6.8.9.10.11.12.14.15.16.17.18} + \&c.\Big) \\ &+ Bx^2\Big(1 + \frac{ax^6}{4.5.6.7.8} + \frac{(ax^6)^2}{4.5.6.7.8.10.11.12.13.14} + \&c.\Big) \\ &+ Cx^4\Big(1 + \frac{ax^6}{6.7.8.9.10} + \frac{(ax^6)^2}{6.7.8.9.10.12.13.14.15.16} + \&c.\Big). \end{aligned} \tag{21}$$

There is only one arbitrary constant here, for A, B and C are given in terms of a by the three conditions named, which as (4) shows, become at at the limit

$$\int_0^\infty y\,dx = \tfrac{1}{2}, \qquad \int_0^\infty x^2y\,dx = 0, \qquad \int_0^\infty x^4y\,dx = 0.$$

Considering, as in the previous case, that it is not necessary to integrate up to infinity, we suppose x_1 to be a sufficiently large finite limit, and get, after integration,

$$\frac{1}{x_1}\int_0^{x_1} ydx = AS_1 + Bx_1^2S_2 + Cx_1^4S_3 = \frac{1}{2x_1},$$
$$\frac{1}{x_1^3}\int_0^{x_1} x^2ydx = AS_4 + Bx_1^2S_5 + Cx_1^4S_6 = 0,$$
$$\frac{1}{x_1^5}\int_0^{x_1} x^4ydx = AS_7 + Bx_1^2S_8 + Cx_1^4S_9 = 0,$$

where S_1, S_2, &c., are the sums of series involving powers of ax_1^6. These three equations determine A, B, C as follows.

$$\left.\begin{aligned} A &= \frac{S_6(S_6S_8 - S_5S_9)}{2x_1[(S_1S_6 - S_3S_4)(S_6S_8 - S_5S_9) - (S_6S_7 - S_4S_9)(S_2S_6 - S_3S_5)]}, \\ B &= -\frac{A}{x_1^2}\left(\frac{S_6S_7 - S_4S_9}{S_6S_8 - S_5S_9}\right), \qquad C = -\frac{A}{x_1^4}\left(\frac{S_4}{S_6}\right) - \frac{B}{x_1^2}\left(\frac{S_5}{S_6}\right). \end{aligned}\right\} \quad (22)$$

Assuming a suitably large number $c = ax_1^6$, we can compute the numerical values of S_1, S_2, &c., and so find A, B and C in terms of x_1 or of a. The above formulas show that A, B and C vary inversely as x_1, x_1^3 and x_1^5 respectively, that is to say, they vary directly as $(-a)^{\frac{1}{6}}$, $(-a)^{\frac{3}{6}}$ and $(-a)^{\frac{5}{6}}$, respectively, since ax_1^6 is supposed constant, and a as found by (20) is essentially negative. We therefore have

$$A = h_1(-a)^{\frac{1}{6}}, \qquad B = h_2A(-a)^{\frac{1}{3}}, \qquad C = h_3A(-a)^{\frac{2}{3}}. \qquad (23)$$

The analogy with (19) is apparent. Since $y = A$ when $x = 0$, the y dimensions of the curve vary as $(-a)^{\frac{1}{6}}$, so that the x dimensions vary directly as the sixth root of b_6 and also as the sixth root of $k+1$. I have not taken the trouble to compute the numerical values of h_1, h_2 and h_3, and to construct the curve for a particular case, but there seems to be nothing impossible about it.

Analogy leads us to presume also, that a similar method might suffice to determine the limiting curve for repeated adjustments with formulas of higher orders, satisfying four or more of the conditions

$$b = 1, \qquad b_2 = 0, \qquad b_4 = 0, \qquad b_6 = 0, \ \&c.$$

All such curves will resemble each other in having but one arbitrary constant or parameter, while the area of the curve is always unity.

It may be noticed that the limiting value of $2b_2$,

$$2b_2 = 2\int_0^\infty x^2ydx,$$

is the definite integral which, in the probability curve, expresses the square of the mean error, or the mean of the squares of all the errors. Likewise the limit of $2b_4$, in the same curve,

$$2b_4 = 2\int_0^\infty x^4ydx,$$

would express the mean of the fourth powers of the errors; and so on.